University of California - Berkeley
Department of Electrical Engineering & Computer Sciences
EE126 Probability and Random Processes
(Spring 2012)

Discussion 2 Notes
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1. Problem 1.31, page 60 in the text
   Communication through a noisy channel. A binary (0 or 1) message transmitted through a noisy communication channel is received incorrectly with probability ϵ_0 and ϵ_1, respectively. Errors in different symbol transmissions are independent. The channel source transmits a 0 with probability p and transmits a 1 with probability $1 - p$.
   (a) What is the probability that a randomly chosen symbol is received correctly?
   (b) Suppose that the string of symbols 1011 is transmitted. What is the probability that all the symbols in the string are received correctly?
   (c) In an effort to improve reliability, each symbol is transmitted three times and the received symbol is decoded by majority rule. In other words, a 0 (or 1) is transmitted as 000 (or 111, respectively), and it is decoded at the receiver as a 0 (or 1) if and only if the received three-symbol string contains at least two 0s (or 1s, respectively). What is the probability that a transmitted 0 is correctly decoded?
   (d) For what values of ϵ_0 is this an improvement over sending a single 0?
   (e) Suppose that the scheme of part (c) is used. What is the probability that a 0 was transmitted given that the received string is 101?

2. **Sudoku.** (The rules of Sudoku can be read at http://en.wikipedia.org/wiki/Sudoku.) Suppose we randomly place 9 1's over a Sudoku board. What is the probability that these 1's satisfy the following constraints:
   (a) No duplicates in any row or column
   (b) No duplicates in any box
   (c) Completely valid Sudoku locations

Solutions:

1. (a)

$$\begin{aligned} P(correct) &= P(correct|0) * P(0) + P(correct|1) * P(1) \\ &= (1-\epsilon_0)p + (1-\epsilon_1) * (1-p) \end{aligned}$$

   (b) Since all the symbols are independent,

$$\begin{aligned} P(1011\ correct) &= P(1\ correct)^3 P(0\ correct) \\ &= (1-\epsilon_1)^3(1-\epsilon_0) \end{aligned}$$

(c)

$$\begin{aligned} P(0 \textit{ decoded correctly}) &= P(000, 001, 010, 100 | 000 \textit{ transmitted}) \\ &= P(000|000) + P(001|000) + P(010|000) + P(100|000) \\ &= (1-\epsilon_0)^3 + 3(1-\epsilon_0)^2\epsilon_0 \end{aligned}$$

(d) This is an improvement when the quantity in part (c) is greater than $1-\epsilon_0$. Algebra yields that this is the case for $\epsilon_0 > \frac{1}{2}$.

(e)

$$\begin{aligned} P(0 \textit{ sent}|101 \textit{ received}) &= \frac{P(0 \textit{ sent})P(101 \textit{ received}|0 \textit{ sent})}{P(0 \textit{ sent})P(101 \textit{ received}|0 \textit{ sent}) + P(1 \textit{ sent})P(101 \textit{ received}|1 \textit{ sent})} \\ &= \frac{p\epsilon_0^2(1-\epsilon_0)}{p\epsilon_0^2(1-\epsilon_0) + (1-p)(1-\epsilon_1)^2\epsilon_1} \end{aligned}$$

2. (a) There are a total of $\binom{81}{9}$ possible layouts. The problem comes down to counting the number of valid layouts. Consider placing a 1 in a valid location in each row starting from the first. There are 9 options in the first row, 8 in the second (since the 1 cannot be placed in the same column as the first row), 7 in the third, and so on. Thus there are 9! total valid layouts.

$$P(valid) = \frac{\# \textit{ valid layouts}}{\# \textit{ total possible layouts}} = \frac{9!}{\binom{81}{9}}$$

(b) Now the total number of valid layouts is simply 9^9 since you can independently choose the 1 in each box to be in any of the 9 locations.

$$P(valid) = \frac{\# \textit{ valid layouts}}{\# \textit{ total possible layouts}} = \frac{9^9}{\binom{81}{9}}$$

(c) As in part (b) we can place the 1's in each box one at a time. This time, however, we have fewer than 9 options in each box since the previous boxes eliminate some of the rows and columns. It can be seen that the number of valid locations are 9,6,3,6,4,2,3,2,1.

$$P(valid) = \frac{\# \textit{ valid layouts}}{\# \textit{ total possible layouts}} = \frac{9 \cdot 6 \cdot 3 \cdot 6 \cdot 4 \cdot 2 \cdot 3 \cdot 2 \cdot 1}{\binom{81}{9}}$$