Natural Splines

- Draw a “smooth” line through several points

A real draftsman’s spline.

Image from Carl de Boor’s webpage.
Natural Cubic Splines

- **Given** $n + 1$ points
  - Generate a curve with $n$ segments
  - Curves passes through points
  - Curve is $C^2$ continuous

- Use cubics because lower order is better...
Natural Cubic Splines

\[ x(u) = \begin{cases} 
  s_1(u) & \text{if } 0 \leq u < 1 \\
  s_2(u - 1) & \text{if } 1 \leq u < 2 \\
  s_3(u - 2) & \text{if } 2 \leq u < 3 \\
  \vdots & \\
  s_n(u - (n - 1)) & \text{if } n - 1 \leq u \leq n 
\end{cases} \]
Natural Cubic Splines

\[ s_i(0) = p_{i-1} \quad i = 1 \ldots n \]
\[ s_i(1) = p_i \quad i = 1 \ldots n \]
\[ s_i'(1) = s_{i+1}'(0) \quad i = 1 \ldots n - 1 \]
\[ s_i''(1) = s_{i+1}''(0) \quad i = 1 \ldots n - 1 \]
\[ s_1''(0) = s_n''(1) = 0 \]
Natural Cubic Splines

\[ s_i(0) = p_{i-1} \quad i = 1 \ldots n \]
\[ s_i(1) = p_i \quad i = 1 \ldots n \]
\[ s'_i(1) = s'_{i+1}(0) \quad i = 1 \ldots n - 1 \]
\[ s''_i(1) = s''_{i+1}(0) \quad i = 1 \ldots n - 1 \]
\[ s''_1(0) = s''_n(1) = 0 \]

\[
\begin{align*}
\text{Total } 4n \text{ constraints}
\end{align*}
\]
Natural Cubic Splines

- Interpolate data points
- No convex hull property
- Non-local support
  - Consider matrix structure...
- $C^2$ using cubic polynomials
B-Splines

- **Goal:** $C^2$ cubic curves with local support
  - Give up interpolation
  - Get convex hull property
- **Build basis by designing “hump” functions**
B-Splines

\[ b(u) = \begin{cases} 
  b_{-2}(u) & \text{if } u_2 \leq u < u_1 \\
  b_{-1}(u) & \text{if } u_1 \leq u < u_0 \\
  b_{+1}(u) & \text{if } u_0 \leq u < u_1 \\
  b_{+2}(u) & \text{if } u_1 \leq u < u_2 
\end{cases} \]

\[ b''_{-2}(u_{-2}) = b'_{-2}(u_{-2}) = b_{-2}(u_{-2}) = 0 \]
\[ b''_{+2}(u_{+2}) = b'_{+2}(u_{+2}) = b_{+2}(u_{+2}) = 0 \]

\[ b_{-2}(u_{-1}) = b_{-1}(u_{-1}) \]
\[ b_{-1}(u_0) = b_{+1}(u_0) \]
\[ b_{+1}(u_{+1}) = b_{+2}(u_{+1}) \]
B-Splines

\[ b(u) = \begin{cases} 
    b_{-2}(u) & \text{if } u_{-2} \leq u < u_{-1} \\
    b_{-1}(u) & \text{if } u_{-1} \leq u < u_{0} \\
    b_{+1}(u) & \text{if } u_{0} \leq u < u_{+1} \\
    b_{+2}(u) & \text{if } u_{+1} \leq u \leq u_{+2} 
\end{cases} \]

\[ b''_{-2}(u_{-2}) = b'_{-2}(u_{-2}) = b_{-2}(u_{-2}) = 0 \quad \leftarrow 3 \text{ constraints} \]
\[ b''_{+2}(u_{+2}) = b'_{+2}(u_{+2}) = b_{+2}(u_{+2}) = 0 \quad \leftarrow 3 \text{ constraints} \]

\[ b_{-2}(u_{-1}) = b_{-1}(u_{-1}) \\
    b_{-1}(u_{0}) = b_{+1}(u_{0}) \\
    b_{+1}(u_{+1}) = b_{+2}(u_{+1}) \]

\[ \text{Repeat for } b' \text{ and } b'' \]
\[ 3 \times 3 = 9 \text{ constraints} \]

**Total 15 constraints** ....... need one more
B-Splines

\[ b(u) = \begin{cases} 
  b_{-2}(u) & \text{if } u_{-2} \leq u < u_{-1} \\
  b_{-1}(u) & \text{if } u_{-1} \leq u < u_0 \\
  b_{+1}(u) & \text{if } u_0 \leq u < u_{+1} \\
  b_{+2}(u) & \text{if } u_{+1} \leq u \leq u_2 
\end{cases} \]

\[ b''(u_{-2}) = b'_-(u_{-2}) = b_{-2}(u_{-2}) = 0 \quad \leftarrow 3 \text{ constraints} \]
\[ b''(u_{+2}) = b'_+(u_{+2}) = b_{+2}(u_{+2}) = 0 \quad \leftarrow 3 \text{ constraints} \]

\[ b_{-2}(u_{-1}) = b_{-1}(u_{-1}) \]
\[ b_{-1}(u_0) = b_{+1}(u_0) \]
\[ b_{+1}(u_{+1}) = b_{+2}(u_{+1}) \quad \leftarrow \text{Repeat for } b' \text{ and } b'' \]
\[ 3 \times 3 = 9 \text{ constraints} \]

\[ b_{-2}(u_{-2}) + b_{-1}(u_{-1}) + b_{+1}(u_0) + b_{+2}(u_{+1}) = 1 \quad \leftarrow 1 \text{ constraint (convex hull)} \]

**Total 16 constraints**
B-Splines
B-Splines
B-Splines
B-Splines
B-Splines

Example with end knots repeated
B-Splines

- Build a curve with overlapping bumps

- Continuity
  - Inside bumps $C^2$
  - Bumps “fade out” with $C^2$ continuity

- Boundaries
  - Circular
  - Repeat end points
  - Extra end points
B-Splines

- **Notation**
  - The basis functions are the $b_i(u)$
  - “Hump” functions are the concatenated function
    - Sometimes the humps are called basis... can be confusing
  - The $u_i$ are the knot locations
  - The weights on the hump/basis functions are control points
B-Splines

- Similar construction method can give higher continuity with higher degree polynomials
- Repeating knots drops continuity
  - Limit as knots approach each other
- Still cubics, so conversion to other cubic basis is just a matrix multiplication
B-Splines

- **Geometric construction**
  - Due to Cox and de Boor
  - My own notation, beware if you compare w/ text

- Let hump centered on \( u_i \) be \( N_{i,4}(u) \)
  - Cubic is order 4

\[ N_{i,k}(u) \]
  - Is order \( k \) hump, centered at \( u_i \)
  - Note: \( i \) is integer if \( k \) is even
  - else \( (i + 1/2) \) is integer
B-Splines

\[ N_{i,j} (u) = \begin{cases} 1 & \text{if } u_i - \frac{1}{2} \leq u < u_i + \frac{1}{2} \\ 0 & \text{else} \end{cases} \]

\[ N_{i,j} (u) = \frac{(u - u_i - k/2) N_{i+1/2,j-1} (u)}{u_{i+k/2-1} - u_{i-k/2}} + \frac{(u_{i+k/2} - u) N_{i+1/2,j-1} (u)}{u_{i+k/2} - u_{i-k/2+1}} \]

Recursive defn.
$N_{2,4}(u)$ aka $N_{0,4}(u)$
abbreviate with $n$

$N_{-1,5,1}(u)$
$N_{-1,5,1}$
$N_{1,5,1}$

Blend $1 \rightarrow 2$
Blend $2 \rightarrow 3$

Term #1
Term #2

* Notice how they fit together!!

$N_{-5,3}(u)$
$N_{-5,3}(u)$

$N_{0,4}(u)$
* Too hard to draw...
NURBS

- Nonuniform Rational B-Splines
  - Basically B-Splines using homogeneous coordinates
  - Transform under perspective projection
  - A bit of extra control
NURBS

\[
p_i = \begin{bmatrix} p_{ix} \\ p_{iy} \\ p_{iz} \\ p_{iw} \end{bmatrix}
\]

\[
x(u) = \frac{\sum_i \begin{bmatrix} p_{ix} \\ p_{iy} \\ p_{iz} \end{bmatrix} N_i(u)}{\sum_i p_{iw} N_i(u)}
\]

- Non-linear in the control points
- The \( p_{iw} \) are sometimes called “weights”