Today

- Transformations in 3D
- Rotations
  - Matrices
  - Euler angles
  - Exponential maps
  - Quaternions
- SIGGRAPH 2005 submissions
- Note: assignment #2 on Wednesday
3D Transformations

- Generally, the extension from 2D to 3D is straightforward
  - Vectors get longer by one
  - Matrices get extra column and row
  - SVD still works the same way
  - Scale, Translation, and Shear all basically the same
- Rotations get interesting

Translations

For 2D:
\[
\tilde{A} = \begin{bmatrix}
1 & 0 & t_x \\
0 & 1 & t_y \\
0 & 0 & 1
\end{bmatrix}
\]

For 3D:
\[
\tilde{A} = \begin{bmatrix}
1 & 0 & 0 & t_x \\
0 & 1 & 0 & t_y \\
0 & 0 & 1 & t_z \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Scales

\[ \tilde{A} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{For 2D} \]

\[ \tilde{A} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{For 3D} \]

(Axis-aligned!)

Shears

\[ \tilde{A} = \begin{bmatrix} 1 & h_{xy} & 0 \\ h_{yx} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{For 2D} \]

\[ \tilde{A} = \begin{bmatrix} 1 & h_{xy} & h_{xz} & 0 \\ h_{yx} & 1 & h_{yz} & 0 \\ h_{zx} & h_{zy} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{For 3D} \]

(Axis-aligned!)
Shears

\[ \tilde{A} = \begin{bmatrix}
1 & h_{xy} & h_{xz} & 0 \\
h_{yx} & 1 & h_{yz} & 0 \\
h_{zx} & h_{zy} & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \]

Shears \( y \) into \( x \)

Rotations

- 3D Rotations fundamentally more complex than in 2D
  - 2D: amount of rotation
  - 3D: amount and axis of rotation
Rotations

- Rotations still orthonormal
- $\det(R) = 1 \neq -1$
- Preserve lengths and distance to origin
- 3D rotations DO NOT COMMUTE!
- Right-hand rule
- Unique matrices

Axis-aligned 3D Rotations

- 2D rotations implicitly rotate about a third out of plane axis
Axis-aligned 3D Rotations

- 2D rotations implicitly rotate about a third out of plane axis

\[ R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \]

\[ R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

Note: looks same as \( \tilde{R} \)

Axis-aligned 3D Rotations

\[ R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \]

\[ R_y = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \]

\[ R_z = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

"Z is in your face"
Axis-aligned 3D Rotations

\[ R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \]

\[ R_y = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \]

\[ R_z = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

Also right handed “Zup”

Also known as “direction-cosine” matrices
Arbitrary Rotations

- Can be built from axis-aligned matrices:
  
  \[ R = R_{\hat{z}} \cdot R_{\hat{y}} \cdot R_{\hat{x}} \]

- Result due to Euler... hence called Euler Angles
- Easy to store in vector
- \( R = \text{rot}(x, y, z) \)
- But NOT a vector.

\[ R = R_{\hat{z}} \cdot R_{\hat{y}} \cdot R_{\hat{x}} \]
Arbitrary Rotations

- Allows tumbling
- Euler angles are non-unique
- Gimbal-lock
- Moving -vs- fixed axes
  - Reverse of each other

Exponential Maps

- Direct representation of arbitrary rotation
- AKA: axis-angle, angular displacement vector
- Rotate $\theta$ degrees about some axis
- Encode $\theta$ by length of vector
  \[ \theta = |\mathbf{r}| \]
Exponential Maps

- Given vector $\mathbf{r}$, how to get matrix $\mathbf{R}$

- Method from text:
  1. rotate about $x$ axis to put $\mathbf{r}$ into the $x$-$y$ plane
  2. rotate about $z$ axis align $\mathbf{r}$ with the $x$ axis
  3. rotate $\theta$ degrees about $x$ axis
  4. undo #2 and then #1
  5. composite together

Exponential Maps

- Vector expressing a point has two parts
  - $\mathbf{X}_\parallel$ does not change
  - $\mathbf{X}_\perp$ rotates like a 2D point
Exponential Maps

$x' = x_{||} + x_{\perp} \sin(\theta) - x_{\perp} \cos(\theta)$

Rodriguez Formula

$x' = \hat{r} (\hat{r} \cdot x) + \sin(\theta) (\hat{r} \times x) - \cos(\theta) (\hat{r} \times (\hat{r} \times x))$

Linear in $x$

Actually a minor variation ...
Exponential Maps

- Building the matrix

\[ x' = (\hat{\mathbf{r}}\hat{\mathbf{r}}^t) + \sin(\theta)(\hat{\mathbf{r}} \times ) - \cos(\theta)(\hat{\mathbf{r}} \times )(\hat{\mathbf{r}} \times )) x \]

\[ (\hat{\mathbf{r}} \times ) = \begin{bmatrix} 0 & -\hat{r}_z & \hat{r}_y \\ \hat{r}_z & 0 & -\hat{r}_x \\ -\hat{r}_y & \hat{r}_x & 0 \end{bmatrix} \]

Antisymmetric matrix
\( (a \times) b = a \times b \)
Easy to verify by expansion

Exponential Maps

- Allows tumbling
- No gimbal-lock!
- Orientations are space within \( \pi \)-radius ball
- Nearly unique representation
- Singularities on shells at \( 2\pi \)
- Nice for interpolation
Exponential Maps

- Why exponential?
- Recall series expansion of $e^x$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Exponential Maps

- Why exponential?
- Recall series expansion of $e^x$
- Euler: what happens if you put in $i\theta$ for $x$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{-\theta^2}{2!} + \frac{-i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

$$= \left(1 + \frac{-\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i \left(\frac{\theta}{1!} + \frac{-\theta^3}{3!} + \cdots\right)$$

$$= \cos(\theta) + i \sin(\theta)$$
Exponential Maps

- Why exponential?

\[
e^{(\hat{r} \times) \theta} = \mathbf{I} + \frac{(\hat{r} \times) \theta}{1!} + \frac{(\hat{r} \times)^2 \theta^2}{2!} + \frac{(\hat{r} \times)^3 \theta^3}{3!} + \frac{(\hat{r} \times)^4 \theta^4}{4!} + \cdots
\]

But notice that: \((\hat{r} \times)^3 = -\hat{r} \times\)

\[
e^{(\hat{r} \times) \theta} = \mathbf{I} + \frac{(\hat{r} \times) \theta}{1!} + \frac{(\hat{r} \times)^2 \theta^2}{2!} - \frac{(\hat{r} \times) \theta^3}{3!} - \frac{(\hat{r} \times)^2 \theta^4}{4!} + \cdots
\]

\[
e^{(\hat{r} \times) \theta} = (\hat{r} \times) \left( \frac{\theta}{1!} - \frac{\theta^3}{3!} + \cdots \right) + \mathbf{I} + (\hat{r} \times)^2 \left( \frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \cdots \right)
\]

\[
e^{(\hat{r} \times) \theta} = (\hat{r} \times) \sin(\theta) + \mathbf{I} + (\hat{r} \times)^2 (1 - \cos(\theta))
\]
Quaternions

- More popular than exponential maps
- Natural extension of $e^{i\theta} = \cos(\theta) + i\sin(\theta)$
- Due to Hamilton (1843)
  - Interesting history
    - Involves “hermaphroditic monsters”

\[ i^2 = j^2 = k^2 = -1 \]

Q = (z₁, z₂, z₃, s) = (z, s)
Q = iz₁ + jz₂ + kz₃ + s

ij = k  ji = −k
jk = i  k j = −i
ki = j  ik = −j
Quaternions

- Multiplication natural consequence of defn.
  \[ q \cdot p = (z_q s_p + z_p s_q + z_p \times z_q, s_p s_q - z_p \cdot z_q) \]

- Conjugate
  \[ q^* = (-z, s) \]

- Magnitude
  \[ ||q||^2 = z \cdot z + s^2 = q \cdot q^* \]

Quaternions

- Vectors as quaternions
  \[ v = (v, 0) \]

- Rotations as quaternions
  \[ r = (\hat{r} \sin \frac{\theta}{2}, \cos \frac{\theta}{2}) \]

- Rotating a vector
  \[ x' = r \cdot x \cdot r^* \]
  \[ \text{Compare to Exp. Map} \]

- Composing rotations
  \[ r = r_1 \cdot r_2 \]
Quaternions

- No tumbling
- No gimbal-lock
- Orientations are “double unique”
- Surface of a 3-sphere in 4D $\|r\| = 1$
- Nice for interpolation

Rotation Matrices

- Eigen system
  - One real eigenvalue
  - Real axis is axis of rotation
  - Imaginary values are 2D rotation as complex number
- Logarithmic formula
  $$(\hat{r} \times) = \ln(R) = \frac{\theta}{2 \sin \theta} (R - R^\top)$$
  $$\theta = \cos^{-1} \left( \frac{\text{Tr}(R) - 1}{2} \right)$$
  Similar formulae as for exponential...
Rotation Matrices

Consider:

\[
\mathbf{RI} = \begin{bmatrix}
  r_{xx} & r_{xy} & r_{xz} \\
  r_{yx} & r_{yy} & r_{yz} \\
  r_{zx} & r_{zy} & r_{zz}
\end{bmatrix} \begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{bmatrix}
\]

- Columns are coordinate axes after transformation (true for general matrices)
- Rows are original axes in original system (not true for general matrices)

Suggested Reading

- Fundamentals of Computer Graphics by Pete Shirley
  - Chapter 5 (still)
  - Rotation stuff in the book is a bit weak... luckily you have these nice slides!