Today

* Forward kinematics
* Inverse kinematics
  * Pin joints
  * Ball joints
  * Prismatic joints
### Forward Kinematics

- Articulated skeleton
  - Topology (what's connected to what)
  - Geometric relations from joints
  - Independent of display geometry
  - Tree structure
    - Loop joints break “tree-ness”

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### Forward Kinematics

- Root body
  - Position set by “global” transformation
  - Root joint
    - Position
    - Rotation
  - Other bodies relative to root
    - *Inboard* toward the root
    - *Outboard* away from root

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Forward Kinematics

- A joint
  - Joint's inboard body
  - Joint's outboard body

- A body
  - Body's inboard joint
  - Body's outboard joint
  - May have several outboard joints
### Forward Kinematics

- A body
  - Body's inboard joint
  - Body's outboard joint
    - May have several outboard joints
  - Body's parent
  - Body's child
    - May have several children

<table>
<thead>
<tr>
<th>Interior joints</th>
</tr>
</thead>
<tbody>
<tr>
<td>Typically not 6 DOF joints</td>
</tr>
<tr>
<td>Pin - rotate about one axis</td>
</tr>
<tr>
<td>Ball - arbitrary rotation</td>
</tr>
<tr>
<td>Prism - translation along one axis</td>
</tr>
</tbody>
</table>
Forward Kinematics

- Pin Joints
  - Translate inboard joint to local origin
  - Apply rotation about axis
  - Translate origin to location of joint on outboard body

Forward Kinematics

- Ball Joints
  - Translate inboard joint to local origin
  - Apply rotation about arbitrary axis
  - Translate origin to location of joint on outboard body

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Forward Kinematics

• Prismatic Joints
  • Translate inboard joint to local origin
  • Translate along axis
  • Translate origin to location of joint on outboard body

Forward Kinematics

• Composite transformations up the hierarchy
Forward Kinematics

- Composite transformations up the hierarchy
Forward Kinematics

• Composite transformations up the hierarchy
Inverse Kinematics

- Given
  - Root transformation
  - Initial configuration
  - Desired end point location

- Find
  - Interior parameter settings
Inverse Kinematics

• A simple two segment arm in 2D

\[
\begin{align*}
    p_z &= l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) \\
    p_x &= l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)
\end{align*}
\]

In逆Kinematics

• Direct IK: solve for the parameters

\[
\begin{align*}
    \theta_2 &= \cos^{-1} \left( \frac{p_z^2 + p_x^2 - l_1^2 - l_2^2}{2l_1l_2} \right) \\
    \theta_1 &= \frac{-p_z l_2 \sin(\theta_2) + p_x (l_1 + l_2 \cos(\theta_2))}{p_x l_2 \sin(\theta_2) + p_z (l_1 + l_2 \cos(\theta_2))}
\end{align*}
\]

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Inverse Kinematics

• Why is the problem hard?
  • Multiple solutions separated in configuration space

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Inverse Kinematics

• Why is the problem hard?
  • Solutions may not always exist

Numerical Solution

• Start in some initial configuration
• Define an error metric (e.g. goal pos - current pos)
• Compute Jacobian of error w.r.t. inputs
• Apply Newton's method (or other procedure)
• Iterate...
Inverse Kinematics

- Recall simple two segment arm:

\[
\begin{align*}
p_x &= l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) \\
p_z &= l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)
\end{align*}
\]

Inverse Kinematics

- We can write of the derivatives:

\[
\begin{align*}
\frac{\partial p_x}{\partial \theta_1} &= l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) \\
\frac{\partial p_z}{\partial \theta_1} &= -l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2) \\
\frac{\partial p_x}{\partial \theta_2} &= -l_2 \sin(\theta_1 + \theta_2) \\
\frac{\partial p_z}{\partial \theta_2} &= +l_2 \cos(\theta_1 + \theta_2)
\end{align*}
\]

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Inverse Kinematics

Direction in Config. Space
\[ \begin{align*}
\theta_1 &= c_1 \theta_* \\
\theta_2 &= c_2 \theta_*
\end{align*} \]

\[ \frac{\partial p_z}{\partial \theta_*} = c_1 \frac{\partial p_z}{\partial \theta_1} + c_2 \frac{\partial p_z}{\partial \theta_2} \]

The Jacobian (of \( p \) w.r.t. \( \theta \))
\[ J_{ij} = \frac{\partial p_i}{\partial \theta_j} \]

Example for two segment arm
\[ J = \begin{bmatrix}
\frac{\partial p_z}{\partial \theta_1} & \frac{\partial p_z}{\partial \theta_2} \\
\frac{\partial p_x}{\partial \theta_1} & \frac{\partial p_x}{\partial \theta_2}
\end{bmatrix} \]
Inverse Kinematics

The Jacobian (of $p$ w.r.t. $\theta$)

$$J = \begin{bmatrix} \frac{\partial p_z}{\partial \theta_1} & \frac{\partial p_z}{\partial \theta_2} \\ \frac{\partial p_x}{\partial \theta_1} & \frac{\partial p_x}{\partial \theta_2} \end{bmatrix}$$

$$\frac{\partial p}{\partial \theta^*_i} = J \cdot \begin{bmatrix} \frac{\partial \theta_1}{\partial \theta^*_i} \\ \frac{\partial \theta_2}{\partial \theta^*_i} \end{bmatrix} = J \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Solving for $c_1$ and $c_2$

$$c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad dp = \begin{bmatrix} dp_z \\ dp_x \end{bmatrix}$$

$$dp = J \cdot c$$

$$c = J^{-1} \cdot dp$$
Inverse Kinematics

Solving for $c_1$ and $c_2$

$\begin{align*}
\theta_1 & \quad l_1 \\
\theta_2 & \quad l_2 \\
\theta_3 & \quad p \\
\end{align*}$

$e = dp$

$\begin{align*}
dp &= J \cdot c \\
c &= J^{-1} \cdot dp
\end{align*}$

In the Jacobian invertible?

Inverse Kinematics

- Problems
  - Jacobian may (will!) not always be invertible
    - Use pseudo inverse (SVD)
    - Robust iterative method
    - Jacobian is not constant
  - Nonlinear optimization, but problem is (mostly) well behaved

$J = 
\begin{bmatrix}
\frac{\partial p_x}{\partial \theta_1} & \frac{\partial p_x}{\partial \theta_2} \\
\frac{\partial p_y}{\partial \theta_1} & \frac{\partial p_y}{\partial \theta_2} \\
\end{bmatrix} = J(\theta)$

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Inverse Kinematics

• More complex systems
  • More complex joints (prism and ball)
  • More links
  • Other criteria (COM or height)
  • Hard constraints (joint limits)
  • Multiple criteria and multiple chains

• Some issues
  • How to pick from multiple solutions?
  • Robustness when no solutions
  • Contradictory solutions
  • Smooth interpolation
    • Interpolation aware of constraints
Inverse Kinematics

**Prism Joints**

\[ p_z = l_1 + d \]
\[ p_x = 0 \]

**Inverse Kinematics**

**Ball Joints**

\[ p = \hat{r} (\hat{r} \cdot x) + \sin(||r||)(\hat{r} \times x) - \cos(||r||)(\hat{r} \times (\hat{r} \times x)) \]

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Inverse Kinematics

Ball Joints (moving axis)

\[ dp = [dr] \cdot e^{[r]} \cdot x = [dr] \cdot p = -[p] \cdot dr \]

That is the Jacobian for this joint

\[
[r] = \begin{bmatrix}
0 & -r_3 & r_2 \\
r_3 & 0 & -r_1 \\
-r_2 & r_1 & 0
\end{bmatrix}
\]

\[ r \cdot x = r \times x \]

Inverse Kinematics

Ball Joints (fixed axis)

\[ dp = (d\theta) \hat{r} \cdot x = -[x] \cdot \hat{r} d\theta \]

That is the Jacobian for this joint

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Inverse Kinematics

• Many links / joints
  • Need a generic method for building Jacobian

\[ \begin{align*}
\mathbf{J} &= \begin{bmatrix} J_3 & J_{2b} & J_{2a} & J_{1b} \end{bmatrix} \\
\mathbf{d} &= \begin{bmatrix} d_3 \\ d_{2b} \\ d_{2a} \\ d_{1b} \end{bmatrix} \\
\mathbf{dp} &\neq \mathbf{J} \cdot \mathbf{dd}
\end{align*} \]
Inverse Kinematics

Transformation from body to world

\[ X_{0\leftarrow i} = \prod_{j=1}^{i} X_{(j-1)\leftarrow j} = X_{0\leftarrow 1} \cdot X_{1\leftarrow 2} \cdots \]

Rotation from body to world

\[ R_{0\leftarrow i} = \prod_{j=1}^{i} R_{(j-1)\leftarrow j} = R_{0\leftarrow 1} \cdot R_{1\leftarrow 2} \cdots \]

Need to transform Jacobians to common coordinate system (WORLD)

\[ J_{i,\text{WORLD}} = R_{0\leftarrow (i-1)} \cdot J_{i} \]
Inverse Kinematics

\[
J = \begin{bmatrix}
R_{0\rightarrow 2b} \cdot J_3(\theta_3, p_3) \\
R_{0\rightarrow 2a} \cdot J_{2b}(\theta_{2b}, X_{2b\rightarrow 3} \cdot p_3) \\
R_{0\rightarrow 1} \cdot J_{2a}(\theta_{2a}, X_{2a\rightarrow 3} \cdot p_3) \\
J_1(\theta_1, X_{1\rightarrow 3} \cdot p_3)
\end{bmatrix}^T
\]

\[
d = \begin{bmatrix}
d_3 \\
d_{2b} \\
d_{2a} \\
d_{1b}
\end{bmatrix}
\]

Note: Each row in the above should be transposed....

\[
dp = J \cdot \dd
\]

Suggested Reading

- Advanced Animation and Rendering Techniques by Watt and Watt
  - Chapters 15 and 16

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