Today
Natural Splines

- Draw a “smooth” line through several points

A real draftsman’s spline.

Image from Carl de Boor’s webpage.

Natural Cubic Splines

- Given \( n + 1 \) points
  - Generate a curve with \( n \) segments
  - Curves passes through points
  - Curve is \( C^2 \) continuous

- Use cubics because lower order is better...
Natural Cubic Splines

\[ x(u) = \begin{cases} 
  s_1(u) & \text{if } 0 \leq u < 1 \\
  s_2(u - 1) & \text{if } 1 \leq u < 2 \\
  s_3(u - 2) & \text{if } 2 \leq u < 3 \\
  \vdots & \\
  s_n(u - (n - 1)) & \text{if } n - 1 \leq u \leq n 
\end{cases} \]

\[ s_i(0) = p_{i-1} \quad i = 1 \ldots n \]
\[ s_i(1) = p_i \quad i = 1 \ldots n \]
\[ s'_i(1) = s'_{i+1}(0) \quad i = 1 \ldots n - 1 \]
\[ s''_i(1) = s''_{i+1}(0) \quad i = 1 \ldots n - 1 \]
\[ s''_1(0) = s''_n(1) = 0 \]

Total 4n constraints
Natural Cubic Splines

- Interpolate data points
- No convex hull property
- Non-local support
  - Consider matrix structure...
- $C^2$ using cubic polynomials

B-Splines

- Goal: $C^2$ cubic curves with local support
  - Give up interpolation
  - Get convex hull property
  - Build basis by designing “hump” functions
\[ b(u) = \begin{cases} 
  b_{-2}(u) & \text{if } u_{-2} \leq u < u_{-1} \\
  b_{-1}(u) & \text{if } u_{-1} \leq u < u_{0} \\
  b_{+1}(u) & \text{if } u_{0} \leq u < u_{+1} \\
  b_{+2}(u) & \text{if } u_{+1} \leq u \leq u_{+2} 
\end{cases} \]

\[ b''_{-2}(u_{-2}) = b''_{-2}(u_{-2}) = b_{-2}(u_{-2}) = 0 \quad \leftarrow 3 \text{ constraints} \]
\[ b''_{+2}(u_{+2}) = b''_{+2}(u_{+2}) = b_{+2}(u_{+2}) = 0 \quad \leftarrow 3 \text{ constraints} \]

\[ b_{-2}(u_{-1}) = b_{-1}(u_{-1}) \]
\[ b_{-1}(u_{0}) = b_{+1}(u_{0}) \]
\[ b_{+1}(u_{+1}) = b_{+2}(u_{+1}) \quad \leftarrow \text{Repeat for } b' \text{ and } b'' \]

3x3=9 \text{ constraints}

Total 15 constraints ...... need one more

\[ b_{-2}(u_{-2}) + b_{-1}(u_{-1}) + b_{+1}(u_{0}) + b_{+2}(u_{+1}) = 1 \quad \leftarrow 1 \text{ constraint (convex hull)} \]

Total 16 constraints
B-Splines

- Build a curve w/ overlapping bumps
- Continuity
  - Inside bumps $C^2$
  - Bumps “fade out” with $C^2$ continuity
- Boundaries
  - Circular
  - Repeat end points
  - Extra end points

B-Splines

- Notation
  - The basis functions are the $b_i(u)$
  - “Hump” functions are the concatenated function
    - Sometimes the humps are called basis... can be confusing
  - The $u_i$ are the knot locations
  - The weights on the hump/basis functions are control points
B-Splines

- Similar construction method can give higher continuity with higher degree polynomials
- Repeating knots drops continuity
  - Limit as knots approach each other
- Still cubics, so conversion to other cubic basis is just a matrix multiplication

Geometric construction
- Due to Cox and de Boor
- My own notation, beware if you compare with text

Let hump centered on $u_i$ be $N_{i,4}(u)$

Cubic is order 4

$N_{i,k}(u)$ is order $k$ hump, centered at $u_i$

Note: $i$ is integer if $k$ is even
else $(i+1/2)$ is integer
B-Splines

\[ N_{i,j}(u) = \begin{cases} 1 & \text{if } u_i - u_{i+1} \leq u < u_{i+1} - u_{i+2} \\ 0 & \text{else} \end{cases} \]

\[ N_{i,j}(u) = \left( u - u_{i-j} \right) N_{i-1,j}(u) \left( \frac{u_{i+j} - u_{i+1}}{u_{i+j} - u_{i+1}} \right) + \left( u_{i+j} - u \right) N_{i+1,j}(u) \left( \frac{u_{i+j} - u_{i+2}}{u_{i+j} - u_{i+2}} \right) \]

Recursive definition.
**NURBS**

- **Nonuniform Rational B-Splines**
  - Basically B-Splines using homogeneous coordinates
  - Transform under perspective projection
  - A bit of extra control

\[
p_i = \begin{bmatrix} p_{ix} \\ p_{iy} \\ p_{iz} \\ p_{iw} \end{bmatrix}
\]

\[
x(u) = \frac{\sum_i \begin{bmatrix} p_{ix} \\ p_{iy} \\ p_{iz} \end{bmatrix} N_i(u)}{\sum_i p_{iw} N_i(u)}
\]

- Non-linear in the control points
- The \( p_{iw} \) are sometimes called “weights”